

A Note on the Topology of Space-time in Special Relativity

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Abstract

We show that a topology can be defined in the four dimensional space-time of special relativity so as to obtain a topological semigroup for time. The Minkowski 4-vector character of space-time elements as well as the key properties of special relativity are still the same as in the standard theory. However, the new topological structure allows the possibility of an intrinsic asymmetry in the time evolution of physical systems.

1 Introduction

It is the received point of view that Einstein's special theory of relativity is above all a theory of time. In some sense, the theory “unifies” space and time into a single entity, a four dimensional space-time, and a physical event is cataloged by a point (x_0, \mathbf{x}) in this space-time. The new notion of time, though quite different from the preceding Newtonian time, still has the reversible, symmetric property of the latter. More precisely, time in relativity is assumed to be modeled by the Euclidean real line, and as such is reversible and symmetric in the sense that the Euclidean real line is a Lie group under addition. Within its topological and algebraic structure, there is no natural

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way to define a flow of time in the four dimensional space-time of special relativity.

On the other hand, the time evolution of most macroscopic systems has an amply manifest irreversible character, often associated with the second law of thermodynamics. Moreover, it is believed that the time evolution of certain microphysical processes, such as resonance scattering and the decay of elementary particles, also possesses such an irreversible nature. Various attempts have been made to develop a quantum theory which accommodates the time asymmetry of such phenomena [1–3], and even the asymmetry in the time evolution of the universe as a whole [4, 5]. Perhaps the most significant feature common to all these developments is that the asymmetry in time evolution is attempted to be obtained as a property of its representation in the space of states of the system. This is particularly notable in [1–3] where the asymmetric time evolution of the microphysical system is realized by way of a *semigroup* of continuous linear operators defined in a suitably constructed rigged Hilbert space.

What is implicitly assumed in these theories is that the time which gets *represented asymmetrically* is still the time of special relativity, and as such, has no manifest irreversibility or asymmetry at the space-time level. That is, the evolution parameter associated with the macroscopic apparatuses which characterize the states and observables appertaining to a quantum physical system is still taken to be the classical time of special relativity, reversible and static; it is the evolution parameter of the states and observables of the quantum physical system that acquires an asymmetry.

The main technical result we present in this paper is a topological structure for the space-time of special relativity that allows for asymmetric time evolutions. Further, this topological structure is introduced in a manner completely consistent with the tenets of special relativity – in particular, without contradicting any of the experimental tests confirming special relativity. However, the new topology provides a better framework for the time-asymmetric quantum theories, such as those developed in [1, 2, 4, 5, 3], in that it endows the structure of a topological semigroup on the set of space-time translations and consequently leads to a *Poincaré semigroup* for relativistic symmetries (and asymmetries).

2 A Topology for Space-time

The space-time of special relativity is assumed to be a four dimensional manifold, M . It has the topology of \mathbb{R}^4 , the four dimensional Euclidean space. A metric tensor $g_{\mu\nu}$ is also defined on M , for the “length” preserved under Lorentz transformations is not that which is compatible with the Euclidean topology. In fact, what is at the heart of all experimentally observable predictions of special relativity is this metric structure, and not the topological structure of M , which is Euclidean.

It is perhaps the case that the mathematical structure of a physical theory, especially its topology, is never completely determined by the physics of the processes it seeks to describe. A topological structure involves such notions as local bases at a point and infinite sequences which are indeterminable by physics alone because the totality of experimental information is never “complete”: The number of experiments that can be performed is finite; the experimental apparatuses allow the measurement of only finite quantities—not infinitesimals; and no measurement is without error. The theoreticians use the freedoms resulting from this incompleteness to construct the mathematical structure of a theory in a way that is not necessarily dictated completely by physics, and how these freedoms are utilized in a given theory is to be valued on the grounds of its overall success. Indeed, the absence of the uniqueness of the topology of a physical theory is an interesting query in its own right. A case in point is the Euclidean topology of M : since it involves open sets (or infinite sequences), just as any other topological structure, this Euclidean topology cannot be deduced from experiments alone. In addition, it is not the most convenient one to use when describing certain phenomena, such as time asymmetric processes. Here, we propose to alter the topological structure of M while leaving intact the (algebraic) properties of how an element of M transforms under Lorentz transformations.

Before we introduce this topology, it is worthwhile to recall a few well known definitions and notions from the theory of groups:

Groups A group is a set G with an operation $G \otimes G \rightarrow G$, denoted by $(a, b) \rightarrow ab$, such that

- a) The operation is associative, i.e., $(ab)c = a(bc)$.
- b) There is an identity element e in G such that $ea = ae = a$ for all $a \in G$.
- c) For every $a \in G$, there exists its inverse a^{-1} such that $aa^{-1} = a^{-1}a = e$.

Notice that this operation, often called multiplication, imposes an *algebraic* structure on the set G ; it is not a topological structure.

Topological Groups A topological group is a group in which a topology is defined so as to make the above group operations continuous. That is, for each $a \in G$, the mappings $x \rightarrow ax$ and $x \rightarrow xa$ are homeomorphisms of G onto G . So is the mapping $x \rightarrow x^{-1}$. These continuity requirements can be more concisely stated by way of the continuity of the mapping $f : G \otimes G \rightarrow G$ defined by

$$f(a, b) = ab^{-1} \quad (2.1)$$

It is clear that such a topology on G is completely determined by any local base at the identity element e of G .

Lie Groups A topological group G is called a Lie group if its topology is such that G is a differentiable manifold and the mapping (2.1) is C^∞ . Many symmetry transformations in physics are assumed to constitute Lie groups; the Poincaré group and symmetry groups of particle physics such as $SU(3)$ are examples. Along with these well known classical concepts, we are in need of the notion of topological semigroup. For the purposes of this paper, we define it as follows:

Topological Semigroups A topological semigroup is a topological space S with an internal operation $S \otimes S \rightarrow S$, denoted by $(a, b) \rightarrow ab$, such that

- a) The operation is continuous.
- b) It is associative, i.e., $(ab)c = a(bc)$.
- c) There is an identity element e in S such that $ea = ae = a$ for all $a \in S$.

Thus every topological group is a topological semigroup; of course, what is of interest here is those semigroups that are not topological groups.

After these introductory remarks, we now introduce the central idea of this paper.

Definition 2.1. *Consider the collection of intervals of the form $[a, b)$, $a, b \in \mathbb{R}$, where \mathbb{R} is the real line. It is clear that these sets provide a base for a topology for \mathbb{R} . Let $\tilde{\mathbb{R}}$ denote the set of real numbers endowed with this topology.*

It is easy to verify that the mapping $\tilde{\mathbb{R}} \otimes \tilde{\mathbb{R}} \rightarrow \tilde{\mathbb{R}}$ defined by $(t_1, t_2) \rightarrow t_1 + t_2$ is continuous. However, the mapping $t \rightarrow -t$ is not continuous on $\tilde{\mathbb{R}}$. This

means $\tilde{\mathbb{R}}$ is a topological semigroup –not a Lie group– under the operation of addition. Further, in contrast to \mathbb{R} , $\tilde{\mathbb{R}}$ is not locally compact.

We want to propose $\tilde{\mathbb{R}}$ as the mathematical image of time. It is interesting to notice that \mathbb{R} is an algebraic group, and thus a notion of past can still be defined by way of the mapping $t \rightarrow -t$. Since this mapping is not continuous, however, $\tilde{\mathbb{R}}$ does not have the reversible (i.e., Lie group) character that time acquires when modeled by the usual Euclidean line \mathbb{R} . Now we may define our space-time:

Definition 2.2. *Consider the direct product space $\tilde{\mathbb{R}} \otimes \mathbb{R}^3$, where \mathbb{R}^3 is the usual three dimensional Euclidean space. Define a topology on $\tilde{\mathbb{R}} \otimes \mathbb{R}^3$ by declaring sets of the form $V_1 \otimes V_2$ open when V_1 is open in $\tilde{\mathbb{R}}$ and V_2 , in \mathbb{R}^3 . Let \mathcal{M} be the space $\tilde{\mathbb{R}} \otimes \mathbb{R}^3$ endowed with this product topology, and let $\tau_{\mathcal{M}}$ denote the topology itself.*

2.1 The Semigroup of Space-time Translations

\mathcal{M} can be made into (an algebraic) vector space of operators acting on itself. To that end, let (a_0, \mathbf{a}) be an element of \mathcal{M} . Then, for all $x \equiv (x_0, \mathbf{x}) \in \mathcal{M}$, the mapping

$$(a_0, \mathbf{a}) : (x_0, \mathbf{x}) \rightarrow (x_0 + a_0, \mathbf{x} + \mathbf{a}) \quad (2.2)$$

defines the desired action on \mathcal{M} . It is clear that (2.2) is $\tau_{\mathcal{M}}$ -continuous on \mathcal{M} . The multiplication on the set of operators $\{(a_0, \mathbf{a})\}$ defined by means of composition

$$(a_0, \mathbf{a})(b_0, \mathbf{b}) = (a_0 + b_0, \mathbf{a} + \mathbf{b}) \quad (2.3)$$

is continuous with respect to $\tau_{\mathcal{M}}$. Furthermore, the inverse mapping

$$(a_0, \mathbf{a}) \rightarrow (a_0, \mathbf{a})^{-1} = (-a_0, -\mathbf{a}) \quad (2.4)$$

is not continuous in the topology $\tau_{\mathcal{M}}$. Therefore, we see that \mathcal{M} acquires the structure of a topological semigroup (of operators on \mathcal{M} itself). We shall refer to this semigroup as the semigroup of space-time translations, or simply as the translation semigroup, \mathcal{T} .

2.2 Lorentz Transformations

Let B be the open unit ball in \mathbb{R}^3 , i.e.,

$$B = \{\mathbf{v} : \mathbf{v} \in \mathbb{R}^3, |\mathbf{v}| < 1\}$$

For every $\mathbf{v} \in B$, we may define a linear operator $\Lambda(\mathbf{v})$ on \mathcal{M} by way of the equality

$$\Lambda(\mathbf{v}) \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma(x_0 - \mathbf{v} \cdot \mathbf{x}) \\ \mathbf{x} + \frac{\gamma-1}{v^2}(\mathbf{v} \cdot \mathbf{x})\mathbf{v} - \gamma\mathbf{v}x_0 \end{pmatrix} \quad (2.5)$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$. As a matrix, the $\Lambda(\mathbf{v})$ has the form

$$\Lambda(\mathbf{v}) = \begin{pmatrix} \gamma & -\gamma v_1 & -\gamma v_2 & -\gamma v_3 \\ -\gamma v_1 & 1 + \frac{\gamma-1}{v^2}v_1^2 & \frac{\gamma-1}{v^2}v_1v_2 & \frac{\gamma-1}{v^2}v_1v_3 \\ -\gamma v_2 & \frac{\gamma-1}{v^2}v_2v_1 & 1 + \frac{\gamma-1}{v^2}v_2^2 & \frac{\gamma-1}{v^2}v_2v_3 \\ -\gamma v_3 & \frac{\gamma-1}{v^2}v_3v_1 & \frac{\gamma-1}{v^2}v_3v_2 & 1 + \frac{\gamma-1}{v^2}v_3^2 \end{pmatrix} \quad (2.6)$$

where $\mathbf{v} = (v_1, v_2, v_3)$. Both (2.5) and (2.6) are well known from the standard theory: $\Lambda(\mathbf{v})$ is just the familiar Lorentz boost operator on the space-time manifold M . Thus, algebraically, the boost operators (defined by (2.5) or (2.6)) on the space-time \mathcal{M} are identical to the those on the conventional space-time M of special relativity. However, the space-time is now endowed with a different topology $\tau_{\mathcal{M}}$, and we must verify that the operators $\Lambda(\mathbf{v})$ are continuous with respect to $\tau_{\mathcal{M}}$.

Recall that an operator A defined on a topological space S is said to be continuous if for every open set U of S there exists another W such that $A(W) \subset U$. Further, in the present case it is sufficient to consider a boost operator of the form $\Lambda(v_1)$, for which (2.5) reduces to

$$\Lambda(v_1)(x_0, \mathbf{x}) = (\gamma(x_0 - v_1x_1), \gamma(x_1 - v_1x_0), x_2, x_3) \equiv (x'_0, \mathbf{x}') \quad (2.7)$$

where $v_1 \equiv \mathbf{v} = (v_1, 0, 0)$. Next, let U be an open neighborhood of (x'_0, \mathbf{x}') of the form $[x'_0, x'_0 + \epsilon) \otimes (x'_1 - \epsilon, x'_1 + \epsilon) \otimes V$, where $V = (x'_2 - \epsilon, x'_2 + \epsilon) \otimes (x'_3 - \epsilon, x'_3 + \epsilon)$. It then follows from (2.7), which shows that $\Lambda(v_1)$ acts on the coordinates x_2 and x_3 as the identity, that any neighborhood W of (x_0, \mathbf{x})

of the form $[x_0, x_0 + \delta) \otimes (x_1 - \delta, x_1 + \delta) \otimes V$, where $\delta < \epsilon \sqrt{\frac{1-v_1}{1+v_1}}$, fulfills the relation

$$\Lambda(v_1)(W) \subset U \quad (2.8)$$

Therefore, $\Lambda(v_1)$ is a continuous operator on \mathcal{M} .

It is obvious that the rotation operators $R(\boldsymbol{\theta})$ are also continuous on $\mathcal{M} = \tilde{\mathbb{R}} \otimes \mathbb{R}^3$ as they act non-trivially only on \mathbb{R}^3 .

Now, let $L = \{\Lambda\}$ be the totality of the boost operators $\Lambda(\mathbf{v})$ and rotation operators $R(\boldsymbol{\theta})$. As in the standard theory, under the multiplication defined by usual composition of operators, L is a Lie group—the well known homogeneous Lorentz group. We have shown that it is a group of continuous operators on the new space-time \mathcal{M} .

2.3 Poincaré Semigroup

Consider the semidirect product of the translation semigroup \mathcal{T} with the Lorentz group L . Following the common practice, we denote elements of this semidirect product set \mathcal{P} by (Λ, a) , where $\Lambda \in L$ and $a \in \mathcal{T}$. From the considerations of Sections 2.1 and 2.2, we see that (Λ, a) is a continuous operator on \mathcal{M} , defined by $(\Lambda, a)x = \Lambda x + a$. As usual, we define a product rule on \mathcal{P} by the composition of operators:

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2) \quad (2.9)$$

This is an associative multiplication on \mathcal{P} under which the set remains closed. Furthermore, for each $(\Lambda, a) \in \mathcal{P}$, there exists an inverse element $(\Lambda, a)^{-1}$ given by,

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a) \quad (2.10)$$

Thus, under the product rule (2.9), \mathcal{P} acquires the structure of an *algebraic* group.

Consider now the topological properties of \mathcal{P} . From (2.3) and (2.8), we see that (2.9) is a continuous mapping of $\mathcal{P} \otimes \mathcal{P}$ into \mathcal{P} . However, (2.4) implies that (2.10) is not continuous on \mathcal{P} . This means that the multiplication defined by (2.9) turns \mathcal{P} into a *topological semigroup* with respect to the new topology $\tau_{\mathcal{M}}$ we have introduced on space-time. Recall that under the usual Euclidean topology, the multiplication (2.9) makes \mathcal{P} a Lie group, the

very well known Poincaré group. We still retain the algebraic structure of the Poincaré group, but introduce here a topology that makes \mathcal{P} only a semigroup. We call this topological semigroup the Poincaré semigroup.

3 Concluding Remarks

That the set of translations and Lorentz transformations on the relativistic space-time forms a Lie group is a mathematical assumption—one perhaps intrinsically incapable of being verified by direct experiments. Among the conclusions to which this assumption leads is the necessarily reversible, unitary time evolution (in the Hilbert space representation) of quantum physical systems. However, as pointed out in the Introduction, there exist many physical processes the time evolution of which is not unitary and reversible. Although attempts have been made to construct quantum physical theories to describe these phenomena, how these theories are to be reconciled with the structure of relativistic space-time (in particular, the Lie group structure of space-time translations) has not been studied.

This brief note investigates the possibility of endowing the four dimensional space-time of special relativity with a topology which is different from its usual Euclidean topology. We still retain the key properties of special relativity, all of which originate from the algebraic structure of Lorentz transformations. The new topology, however, allows us to view time in way that is quite different from the static character it assumes in orthodox special relativity. As immediate consequences of the new topological structure, we have shown that space-time translations on \mathcal{M} define a topological semigroup, whereupon we obtained the Poincaré semigroup for the set of relativistic transformations on \mathcal{M} . This structure may be the one that provides a proper context and framework for the time asymmetric quantum theories, such as those developed in some of the works cited below. It now remains to investigate the implications of this topology on the fundamental equations of physics—i.e., the meaning of partial derivatives with respect to time needs to be explored—and to construct the representations of the Poincaré semigroup in the space of states of quantum mechanical systems. We shall undertake these tasks in a forthcoming paper.

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